

# Heat Transfer in Superfluids: Effect of Gravity

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## Abstract

We discuss the influence of an external field on energy transport in superfluid. He-II is not isothermal in presence of Earth gravity; instead, it supports finite temperature gradient given by a Fourier-like equation. We calculate asymptotic behavior of the effective heat resistance in the vicinity of the  $\lambda$ -transition.

## 1 INTRODUCTION

Superfluid is known to be (and it was originally referred to as) a *super-heat-conductor*. The heat in superfluid is transferred by the normal flow. This is fundamentally different [1] from the Fourier law thermal conduction in ordinary liquids. Instead, it has a nature of *convection*. In uniform superfluid the flow (and the heat transfer) is dissipationless, *i.e.*, it happens at constant temperature.

Experimentally observed dissipation in bulk uniform superfluid (*i.e.*, when the friction with walls is neglected) is believed to be due to the vortex creation. This scenario leads to the temperature gradients with complicated nonlinear dependence on the supported heat flux. We show that gravity can also explain this dissipation.

Earth gravity, like any external field, destroys the system homogeneity: it produces a pressure gradient across the sample. This inhomogeneity is a cause of the normal flow dissipation. In other words, gravity is a friction mechanism between the normal flow and Earth. Due to the friction, the liquid holds a finite temperature gradient whose leading term is proportional to the energy flux. Unlike natural convection in ordinary fluids, this scenario of energy transfer is described by a Fourier-like equation.

To investigate the effect quantitatively one needs to correct superfluid hydrodynamic equations by adding external field. In Sec.2 we write down these equations. The form of dissipative terms however is not self-evident. To ascertain correct appearance of the equations we use Landau-Khalatnikov superfluid hydrodynamic theory in a free-fall frame of reference. Translating all terms to the laboratory frame we unambiguously get sought equations.

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In Sec.3 we apply obtained equations to the problem of energy transfer in superfluid. For simplicity we restrict ourselves to one-dimensional vertical setup where influence of walls is neglected and the energy goes either up or down. With the help of linearized hydrodynamic equations we find temperature gradient induced by the heat flux. In linear regime the flux and the gradient are proportional to each other. We conclude the section by an expression for the effective heat resistance of this system.

Effective heat resistance derived in Sec.3 becomes larger in the vicinity of the critical temperature. In Sec.4 we show that it diverges at  $\lambda$ -point and calculate its critical behavior.

## 2 HYDRODYNAMICS

Ordinary superfluid hydrodynamic equations are modified by external fields. To find out the effect of gravity we use the equivalence principle of general relativity. For the uniform gravity field of Earth we define a non-inertial free-fall frame of reference  $\tilde{K}$ . It has constant acceleration  $\mathbf{g}$  with respect to the laboratory frame  $K(t, x^i)$ . We also demand that at  $t = 0$  this frame has zero velocity relative to  $K$ . This ensures that hydrodynamic variables are equal in two frames. In the frame  $\tilde{K}$  no gravity is present and (in Newtonian limit) the superfluid is governed by conventional equations [2]

$$\left\{ \begin{array}{l} \frac{\partial \tilde{\rho}}{\partial t} = -\frac{\partial \tilde{j}^i}{\partial x^i} \\ \frac{\partial \tilde{j}^k}{\partial t} = -\frac{\partial \tilde{\Pi}^{ik}}{\partial x^i} \\ \frac{\partial \tilde{v}_s^k}{\partial t} = -\frac{\partial \tilde{\Phi}}{\partial x^k} \\ \frac{\partial \tilde{E}}{\partial t} = -\frac{\partial \tilde{Q}^i}{\partial x^i}, \end{array} \right. \quad (1)$$

where tilde attributes quantities to  $\tilde{K}$ ,  $\rho$  is the fluid density,  $\mathbf{j}$  is the mass flux,  $\mathbf{v}_s$  is the superfluid flow velocity, and  $E$  is the energy density. The fluxes in (1) are given by

$$\tilde{\Pi}^{ik} = v_s^i j^k + v_n^k j^i - \rho v_n^k v_s^i + p \delta_{ik} + \tilde{\pi}^{ik}, \quad (2)$$

$$\tilde{\pi}^{ik} = -\eta \left( \frac{\partial v_n^i}{\partial x^k} + \frac{\partial v_n^k}{\partial x^i} - \frac{2}{3} \frac{\partial v_n^l}{\partial x^l} \delta^{ik} \right) - \delta^{ik} \left( \zeta_1 \frac{\partial (j^i - \rho v_n^i)}{\partial x^i} + \zeta_2 \frac{\partial v_n^l}{\partial x^l} \right), \quad (3)$$

$$\tilde{\Phi} = \mu + v_s^2/2 + \tilde{\phi}, \quad (4)$$

$$\tilde{\phi} = -\zeta_3 \frac{\partial (j^i - \rho v_n^i)}{\partial x^i} - \zeta_1 \frac{\partial v_n^i}{\partial x^i}, \quad (5)$$

and

$$\tilde{Q}^i = \left( \mu + \frac{v_s^2}{2} \right) j^i + ST v_n^i + v_n^i v_n^k (j^k - \rho v_s^k) + (j^i - \rho v_n^i) \tilde{\phi} + v_n^k \tilde{\pi}^{ik} - \kappa \frac{\partial T}{\partial x^i}. \quad (6)$$

Here  $\mathbf{v}_n$  is the velocity of the normal flow,  $p$  is the pressure,  $\mu$  is the chemical potential,  $S$  is the entropy density,  $T$  is the temperature, and  $\eta, \zeta_1, \zeta_2, \zeta_3, \kappa$  are kinetic coefficients. As usual, we took not all the dissipative terms that are in principle possible, but only the largest. In general the dissipation in isotropic superfluid is described by 13 kinetic coefficients rather than by 5 (A. Clark 1963, see discussion in Ref.[1], p.525).

To rewrite the equations in laboratory frame of reference one must translate left-hand side terms in (1) to  $K$ . We finally get

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial j^i}{\partial x^i} = 0 \\ \frac{\partial j^k}{\partial t} + \frac{\partial \tilde{\Pi}^{ik}}{\partial x^i} = \rho g^k \\ \frac{\partial v_s^k}{\partial t} + \frac{\partial \tilde{\Phi}}{\partial x^k} = g^k \\ \frac{\partial E}{\partial t} + \frac{\partial \tilde{Q}^i}{\partial x^i} = j^k g^k. \end{cases} \quad (7)$$

### 3 EFFECTIVE HEAT RESISTANCE

For the steady-state flow all time derivatives in (7) vanish. Consider one-dimensional system where all velocities and fluxes are vertical. Let  $z$ -axis run down along  $\mathbf{g}$ . In typical heat transfer experiment [3] the energy is carried by the superfluid counterflow, where normal and superfluid velocities are directed oppositely to each other to keep zero net mass flux. The condition  $j = 0$  is satisfied along the height of the cell. Under these circumstances the first equation in (7) is trivially satisfied and the other three have the form

$$\begin{cases} \Pi' = \rho g \\ \phi' = g \\ Q' = 0, \end{cases} \quad (8)$$

where dashes denote the  $\partial/\partial z$  derivative.

Within linear (on velocity and temperature gradient) approximation the fluxes in (8) are given by

$$\Pi = p + \zeta_1 (\rho v_n)' - \zeta_2 v_n' - \frac{4}{3} \eta v_n', \quad (9)$$

$$\phi = \mu + \zeta_3 (\rho v_n)' - \zeta_1 v_n', \quad (10)$$

$$Q = TSv_n - \kappa T'. \quad (11)$$

Keeping lowest nontrivial gradient terms (thus assuming the effect of  $\mathbf{g}$  to be small) and excluding  $\mu'$  by means of thermodynamic identity  $\rho d\mu = dp - S dT$ , we obtain from (8), (9) and (10)

$$A'v'_n + Bv_n + ST' = 0,$$

where we introduced

$$A = 2\rho\zeta_1 - \frac{4}{3}\eta - \zeta_2 - \rho^2\zeta_3,$$

$$B = (\zeta'_1 - \rho\zeta'_3)\rho' + (\zeta_1 - \rho\zeta_3)\rho''.$$

Finally employing (8) and (11) we obtain a Fourier-like equation

$$T' = -\varrho Q, \quad (12)$$

where the effective heat resistance  $\varrho$  is determined by

$$\varrho = \frac{1}{TS^2} \left( B - \frac{S'}{S} A' \right). \quad (13)$$

We see that in linear approximation the temperature gradient is proportional to the heat flux and does not depend on its direction. The effective heat resistance is of the second order in system inhomogeneity, *i.e.*,  $\varrho \propto g^2$ . If higher terms are taken into account  $\varrho$  will be different when heated from above or from below.

## 4 CRITICAL BEHAVIOR

The effect discussed in previous section will be present in any Earth-based energy transport experiment, but it seems to be beyond experimental accuracy unless measured in the vicinity of the  $\lambda$ -point. To find critical behavior of the effective heat resistance we need to estimate kinetic coefficients entering (13).

This can be done using the ideas of dynamic scaling [4]. The specific heat index for helium is very small [5], we further let it equal to zero. The superfluid density  $\rho_s$  follows Josephson scaling relation [6]  $\rho_s \propto r_c^{-1} \propto t^{2/3}$ , where  $r_c$  is the correlation length and  $t$  is the reduced temperature  $t = T_\lambda - T$ .\*

Second sound damping [2] at the boundary of the critical region is inversely proportional to the correlation length

$$r_c^{-1} \propto \text{Im} \frac{\omega}{u_2} = \frac{\omega^2}{2\rho u_2^3} \left( \frac{\rho_s}{\rho_n} \left( \frac{4}{3}\eta + \zeta_2 + \rho^2\zeta_3 - 2\rho\zeta_1 \right) + \frac{\kappa\rho}{T} \frac{\partial T}{\partial S} \right). \quad (14)$$

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\*The same letter  $t$  is used to represent the time and the reduced temperature. This should not lead to confusion however: the meaning is clear from the context.

Here  $\rho_n = \rho - \rho_s$ ,  $\omega$  and  $u_2$  are the complex frequency and the velocity of the second sound respectively. The latter is determined by the expression

$$u_2 = \frac{S}{\rho^{1/2}} \sqrt{\frac{\partial T}{\partial S} \frac{\rho_s}{\rho_n}} \propto t^{1/3}.$$

Further assuming that at the critical region boundary all terms in (14) make contribution of the same order to the second sound damping we get for the kinetic coefficients

$$\eta \propto \zeta_1 \propto \zeta_2 \propto \zeta_3 \propto r_c u_2 \rho_s^{-1} \propto t^{-1}. \quad (15)$$

To find the main term of the space derivatives in (13) we note that

$$\frac{\partial}{\partial z} \propto \rho g \left( \frac{\partial}{\partial p} \right)_T \propto -\frac{\rho g}{T} \frac{dT_\lambda}{dp} \left( \frac{\partial}{\partial t} \right)_p.$$

Here the derivative  $dT_\lambda/dp$  is taken along the phase transition curve. Substituting this in (13) and using (15) we get<sup>†</sup>

$$\varrho \propto \frac{g^2}{t^2}. \quad (16)$$

Having shown that finite temperature gradient in superfluid may exist, it is instructive to find the maximum gradient possible. Obtained results on effective heat resistance (being derived within linear approximation) correspond to small temperature gradient and velocities. They are therefore not applicable to the critical velocities region. We will try, however, to use them as an estimate beyond the limits. It is shown in Ref. [8] that the critical energy flux  $Q_c$  in the vicinity of the  $\lambda$ -transition behaves roughly as  $Q_c \propto t^{4/3}$ . Using (16) we see that the maximum temperature gradient  $T'_c$  increases when the temperature approaches  $T_\lambda$  as  $T'_c \propto t^{-2/3}$ .

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<sup>†</sup>It should be emphasized that our accuracy does not allow to resolve between  $T_\lambda$  and  $T_c(Q)$  introduced by some investigators [7]. In linear approximation these two temperature are identical.

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